INVARIANT THEORY AND THE LAMBDA ALGEBRA

BY

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ABSTRACT. Let A be the Steenrod algebra over the field F_2. In this paper we construct for any left A-module M a chain complex whose homology groups are isomorphic to the groups $\mathrm{Tor}_s^A(F_2, M)$. This chain complex in homological degree s is built from a ring of invariants associated with the action of the linear group $GL_s(F_2)$ on a certain algebra of Laurent series. Thus, the homology of the Steenrod algebra (and so the Adams spectral sequence for spheres) is seen to have a close relationship to invariant theory. A key observation in our work is that the Adem relations can be described in terms of the invariant theory of $GL_2(F_2)$. Our chain complex is not new: it turns out to be isomorphic to the one constructed by Kan and his coworkers from the dual of the lambda algebra. Thus, one effect of our work is to give an invariant-theoretic interpretation of the lambda algebra. As a consequence we find that the dual of lambda supports an action of the Steenrod algebra that commutes with the differential. The differential itself appears as a kind of "residue map". We are also able to describe the coalgebra structure of the dual of lambda using our invariant-theoretic language.

1. Introduction. Our object in this paper is to use classical invariant theory to describe a chain complex, from which one can compute the "homology groups of the Steenrod algebra", that is, the groups $\mathrm{Tor}_s^A(F_2, M)$, for an arbitrary left A-module M. Our work provides an invariant-theoretic description of the "lambda algebra", introduced by the six authors of [1]. We show how, as a consequence of our work, we can endow lambda in each homological degree with an action of the Steenrod algebra that commutes with the differential.

In more detail, let $P_s = F_2[t_1, \ldots, t_s]$ be the polynomial algebra over F_2 on generators t_i of dimension one. The linear group $GL_s \equiv GL_s(F_2)$ acts on P_s as a group of algebra automorphisms, and the invariant ring $P_s^{GL_s}$ has been computed by Dickson [3]. He found it to be a polynomial algebra on certain generators $Q_{s,i}$ of dimension $2^s - 2^i$:

$$(1.1) \qquad P_s^{GL_s} = F_2[Q_{s,0}, Q_{s,1}, \ldots, Q_{s,s-1}].$$

For applications to the Steenrod algebra we introduce a ring of invariants somewhat larger than (1.1). Let Φ_s be the algebra obtained from P_s by inverting all

nonzero linear forms in the variables t_i; that is, all elements of the type

$$(1.2) \qquad \left\{ \sum_{i=1}^{s} \lambda_i t_i \mid \lambda_i \in F_2; \ \lambda_i \neq 0 \text{ for at least one } i \right\}.$$

Then GL_s acts on Φ_s as a group of algebra automorphisms. We will define a certain GL_s-subalgebra $\Phi_s^+ \subset \Phi_s$ consisting of elements whose denominators are "not too large", in a sense that we make precise in §2. We write

$$(1.3) \qquad \Gamma_s^+ = \left(\Phi_s^+ \right)^{GL_s}$$

for the ring of invariants. Like (1.1), this ring can easily be described in terms of the generators $Q_{s,i}$. We need only let $Q_{s,0}$ appear with negative exponents (see Proposition 2.4).

The main work of this paper is to associate to any left A-module M a certain chain complex $\Gamma^+ M$. As a vector space,

$$(1.4) \qquad \left(\Gamma^+ M \right)_s = \Phi_s^+ \otimes M.$$

The most important property of the chain complex is given by

THEOREM 1.1. *There is a natural isomorphism*

$$(1.5) \qquad \operatorname{Tor}_s^A (F_2, M) = H_s(\Gamma^+ M).$$

Thus, the "homology groups of the Steenrod algebra" are seen to be closely related to the invariant theory of the general linear groups. These results are all obtained in the first four sections of the paper.

Having said this we must confess that only the invariant-theoretic description of our chain complex $\Gamma^+ M$ is new; the complex itself is not. $\Gamma^+ M$ is just the dual of the cochain complex constructed in [1] from the "lambda algebra". In particular if $M = F_2$ we will show how to give $\Gamma^+ F_2$ the structure of a differential coalgebra, in such a way that the following theorem holds.

THEOREM 1.2. *Let Λ^* be the dual of Λ. Then there is an isomorphism of differential coalgebras*

$$(1.6) \qquad \Gamma^+ F_2 \simeq \Lambda^*.$$

We demonstrate this in §8. The effect of our work is thus to give an invariant theoretic interpretation of the lambda algebra.

A close relationship between the Adem relations and the invariant theory of GL_2 is central to the construction of $\Gamma^+ M$, and to the isomorphism (1.6). We describe this connection in §3, and use it prove our main theorems.

By recasting the lambda algebra in invariant-theoretic language, we discover some heretofore unsuspected properties of its dual. We show in §5:

THEOREM 1.3. *Let M be a graded left A-module. For each $s \geqslant 0$, $(\Gamma^+ M)_s$ has a natural left A-action, in such a way that the differential $\partial: (\Gamma^+ M)_s \to (\Gamma^+ M)_{s-1}$ is an A-linear map. The induced A-action on $\operatorname{Tor}_s^A(F_2, M) = H_s(\Gamma^+ M)$ is trivial.*

This result gives an interesting interpretation of the differential. The A-action on $(\Gamma^+ M)_s$ is very rich. The differential kills it.

To describe the A-action on $(\Gamma^+ M)_s$ in general requires that we use the main construction of [8]; but in the special case $M = F_2$ the action on $(\Gamma^+ F_2)_s = \Gamma_s^+$ is easily described. In fact the polynomial algebra P_s acquires an A-action from its identification with the cohomology ring $H^*(RP^\infty \times RP^\infty \times \cdots \times RP^\infty)$. This action extends uniquely to an action on the localization Φ_s^+ (in the manner of [11, Proposition 2.1]) which commutes with the action of GL_s. Consequently, $\Gamma_s^+ = (\Phi_s^+)^{GL_s}$ acquires the structure of an algebra over A, and the action on $(\Gamma^+ M)_s$ described in Theorem 1.3 reduces to this one in the case $M = F_2$.

As another corollary of our invariant-theoretic description of the lambda algebra, we find that the differential on Λ^* can be described in an interesting way. In fact, we show in §9 that $\partial\colon \Gamma_s^+ \to \Gamma_{s-1}^+$ is a kind of residue map. Given $\gamma \in \Gamma_s^+$, one expands γ (a rational function in $t_1, t_2, \ldots, t_s$) as a Laurent series about the origin of the t_s-plane. Then $\partial\gamma$ is the coefficient of t_s^{-1} in the series expansion. We express this result in the form

THEOREM 1.4. *Let* $\gamma \in \Gamma_s^+ = (\Gamma^+ F_2)_s$ *be given. Then*

$$(1.7) \qquad\qquad\qquad \partial\gamma = \mathrm{Res}\,\gamma.$$

Care has been taken to make this paper reasonably selfcontained. In the first four sections we carry out from scratch the construction of $\Gamma^+ M$ and the proof of Theorem 1.1. Comparison with the lambda algebra and the proof of Theorem 1.2 are postponed to §§6–8, and even there prior knowledge of the lambda algebra is not assumed. The proof of Theorem 1.4 is contained in §9. It, too, can be understood entirely within the context of the present paper. Only in §5, where Theorem 1.3 is proved, must we summarize and use the main results of [8].

Although this paper is independent of [9], I express my warmest thanks to William Dwyer and Haynes Miller for pointing out the relationship between the constructions of [9] and the lambda algebra, thus giving the current paper its start. I thank Clarence Wilkerson for suggesting that invariant rings should play a role, and for his help in understanding the coproduct in $\Gamma^+ F_2$. The elegant definition (6.4) of the differential in the lambda algebra was shown to me by Peter Bousfield and Stewart Priddy.

2. Invariant theory. In this section we collect those results from the theory of invariants that we will need to describe the chain complexes $\Gamma^+ M$.

Write $P_s = F_2[t_1, \ldots, t_s]$ for the polynomial algebra over F_2 on s generators, each of dimension one. The group $GL_s \equiv GL_s(F_2)$ acts naturally on the vector space of one-dimensional elements of P_s. We extend this action to the full polynomial ring by requiring GL_s to act as a group of algebra automorphisms.

Let $T_s \subseteq GL_s$ be the subgroup consisting of all upper triangular matrices with "ones" on the main diagonal. The invariant ring $P_s^{T_s}$ has been determined by Mui [7, Theorem 3.4]. He finds that $P_s^{T_s}$ is a polynomial algebra

$$(2.1) \qquad\qquad\qquad P_s^{T_s} = F_2[V_1, V_2, \ldots, V_s]$$

on elements V_k of dimension 2^{k-1}. V_k is given by

$$(2.2) \qquad\qquad V_k = \prod_\lambda (\lambda_1 t_1 + \cdots + \lambda_{k-1} t_{k-1} + t_k)$$

where the product is over all $(k-1)$-tuples $\lambda = (\lambda_1,\ldots,\lambda_{k-1})$ with $\lambda_i \in F_2$.

The invariant ring $P_s^{GL_s}$ has been determined by Dickson [3, pp. 76, 80]. He finds that $P_s^{GL_s}$ is also a polynomial algebra

$$(2.3) \qquad P_s^{GL_s} = F_2[Q_{s,0}, Q_{s,1},\ldots,Q_{s,s-1}]$$

on generators $Q_{s,i}$ of dimension $2^s - 2^i$. The element we call $Q_{s,0}$ is called L_s by Dickson, it is given by

$$(2.4) \qquad Q_{s,0} = \prod_\lambda (\lambda_1 t_1 + \cdots + \lambda_s t_s)$$

where the product is over all s-tuples $\lambda = (\lambda_1,\ldots,\lambda_s)$ with $\lambda_i \in F_2$, for which $\lambda \neq (0,0,\ldots,0)$. Clearly,

$$(2.5) \qquad Q_{s,0} = V_1 V_2 \cdots V_s.$$

The elements $Q_{s,i}$ for $0 < i < s$ are a little harder to describe. Dickson gives $Q_{s,i}$ as a quotient $Q_{s,i} = [0, 1,\ldots,\hat{i},\ldots,s]/Q_{s,0}$, where $[0, 1,\ldots,\hat{i},\ldots,s]$ is a certain determinant in the variables t_i [3, p. 75]. Of course one also has $P_s^{GL_s} \subseteq P_s^{T_s}$, so one can expect a formula for $Q_{s,i}$ in terms of the V_k of (2.2). A formula of this kind is given by Dickson: he describes $Q_{s,i}$ inductively in s by proving

$$(2.6) \qquad Q_{s,i} = V_s \cdot Q_{s-1,i} + Q_{s-1,i-1}^2 \qquad (0 \leqslant i < s).$$

This is [3, equation (4), p. 80]. To interpret (2.6) read $Q_{s-1,s-1} = 1$ if $s \geqslant 1$, and $Q_{s,i} = 0$ if $i < 0$ or $i > s$. Then (2.6) determines the $Q_{s,i}$, and the reader may take it as the definition if he wishes. For example, $Q_{1,0} = V_1 = t_1$; $Q_{2,0} = V_1 V_2 = t_1^2 t_2 + t_1 t_2^2$; $Q_{2,1} = V_1^2 + V_2 = t_1^2 + t_1 t_2 + t_2^2$. Notice that (2.6) implies (2.5).

For our applications to the lambda algebra we will need analogues of (2.1) and (2.3) for the case in which the polynomial algebra P_s is replaced by an algebra of rational functions in the t_i. In fact, let $S(s) \subseteq P_s$ be the multiplicative subset generated by all the one-dimensional classes in (1.2). Let Φ_s be the localization

$$(2.7) \qquad \Phi_s = (P_s)_{S(s)}$$

as in §1. Then GL_s acts on Φ_s as a group of algebra automorphisms. We write $\Delta_s = (\Phi_s)^{T_s}$ and $\Gamma_s = (\Phi_s)^{GL_s}$ for the invariant rings for each $s \geqslant 1$; and we define $\Delta_0 = \Gamma_0 = F_2$. Then from (2.1), (2.3) and (2.4) we obtain easily for each $s \geqslant 1$

$$(2.8) \qquad \Delta_s = (\Phi_s)^{T_s} = F_2[V_1^{\pm 1}, V_2^{\pm 1},\ldots,V_s^{\pm 1}],$$

$$(2.9) \qquad \Gamma_s = (\Phi_s)^{GL_s} = F_2[Q_{s,0}^{\pm 1}, Q_{s,1},\ldots,Q_{s,s-1}].$$

We will need an alternate set of generators for Δ_s. Set

$$(2.10) \qquad v_1 = V_1, \quad v_k = V_k/V_1 V_2 \cdots V_{k-1} \qquad (k \geqslant 2)$$

so that

$$(2.11) \qquad V_k = v_1^{2^{k-2}} v_2^{2^{k-3}} \cdots v_{k-1} v_k \qquad (k \geqslant 2).$$

Then (2.8) could just as well be written

$$(2.12) \qquad \Delta_s = (\Phi_s)^{T_s} = F_2[v_1^{\pm 1}, v_2^{\pm 1},\ldots,v_s^{\pm 1}].$$

The new generators will be useful in constructing the chain complexes $\Gamma^+ M$, and in relating $\Gamma^+ F_2$ to the dual of lambda (see (7.1)). Note that dim $v_k = 1$ for all k.

We will use our new generators to make a coalgebra from the spaces Δ_s. In fact, for any pair of nonnegative integers p, q, for which $p + q = s$ we define an algebra isomorphism $\psi_{p,q}\colon \Delta_s \to \Delta_p \otimes \Delta_q$ by

$$(2.13) \qquad \psi_{p,q}(v_i) = \begin{cases} v_i \otimes 1, & 1 \le i \le p, \\ 1 \otimes v_{i-p}, & p + 1 \le i \le s. \end{cases}$$

Here we understand $\Delta_0 = F_2$; $\psi_{s,0}(x) = x \otimes 1$; $\psi_{0,s}(x) = 1 \otimes x$. If we now set $\Delta = \bigoplus_{s \ge 0}(\Delta_s)$ then the maps (2.13) combine to define a coproduct $\psi\colon \Delta \to \Delta \otimes \Delta$ under which Δ becomes a connected, graded coalgebra.

Similarly let us write $\Gamma = \bigoplus_{s \ge 0}(\Gamma_s)$. We would like to show that Γ is a subcoalgebra of Δ. To this end, we assert that if $p, q \ge 0$ and $p + q = s$ then

$$(2.14) \qquad \psi_{p,q}(Q_{s,i}) = \sum_{j \ge 0} Q_{p,0}^{2^q - 2^j} Q_{p,i-j}^{2^j} \otimes Q_{q,j}$$

for each i, $0 \le i < s$. (The conventions of (2.6) are in force here as well: $Q_{i,i} = 1$ if $i \ge 0$; $Q_{s,i} = 0$ if $i < 0$ or $i > s$.) The proof of (2.14) is an easy induction on s: one uses (2.5), (2.6), (2.11), and the fact that $\psi_{p,q}$ preserves products. The reader may wish to compare (2.14) to Madsen's formula for the diagonal in the dual of the Dyer-Lashof algebra; see Proposition 3.4 of [5]. The formulas are identical.

Since $\psi_{p,q}$ is a map of algebras, (2.14) determines $\psi_{p,q}$ on Γ_s. We have at once

PROPOSITION 2.1. $\psi_{p,q}(\Gamma_s) \subseteq \Gamma_p \otimes \Gamma_q$; so Γ is a subcoalgebra of Δ.

In order to construct the chain complexes $\Gamma^+ M$, we need to single out a certain subspace of Γ_s consisting of those GL_s-invariant rational functions in the variables t_i whose denominators are "not too large". We make this notion precise in the following way. An element σ of the multiplicative subset $S(s)$ can be written uniquely up to order in the form $\sigma = \sigma_1^{i_1} \cdots \sigma_k^{i_k}$, where $i_1, \ldots, i_k \ge 0$ and $\sigma_1, \ldots, \sigma_k$ are distinct one-dimensional elements of P_s. Define the "weight" of σ, written $\rho(\sigma)$, by setting $\rho(\sigma) = \max(i_1, \ldots, i_k)$. Now define the subalgebra $\Phi_s^+ \subseteq \Phi_s$ by

$$(2.15) \qquad \Phi_s^+ = \mathrm{Span}\left\{ \gamma = \beta/\sigma \,\middle|\, \begin{array}{l} \beta \in P_s, \sigma \in S(s) \\ \rho(\sigma) \le \dim \gamma \end{array} \right\}$$

for all $s \ge 1$; and if $s = 0$ write $\Phi_0^+ = F_2$. Note in particular that all elements of Φ_s^+ have nonnegative dimension. A useful lemma in working with Φ_s^+ is

LEMMA 2.2. Let $s \ge 1$. If $\gamma \in \Phi_s^+$ is a homogeneous element then it can be written

$$(2.16) \qquad \gamma = \beta/Q_{s,0}^n$$

where $\beta \in P_s$ and n is an integer for which $0 \le n \le \dim \gamma$.

This is an easy consequence of (2.4) and (2.15).

Clearly the subalgebra $\Phi_s^+ \subseteq \Phi_s$ is mapped to itself by the elements of GL_s. Write

$$(2.17) \qquad \Delta_s^+ = (\Phi_s^+)^{T_s}, \quad \Gamma_s^+ = (\Phi_s^+)^{GL_s}$$

for the invariant rings if $s \geq 1$; and $\Delta_0^+ = \Gamma_0^+ = F_2$. Then Δ_s^+ and Γ_s^+ are subalgebras of Δ_s and Γ_s, respectively. Using Lemma 2.2 we can easily describe monomial bases for these subalgebras.

PROPOSITION 2.3. *If $\gamma \in \Delta_s^+$ then γ can be written as a sum of monomials $v_1^{j_1} \cdots v_s^{j_s}$, each of which lies in Δ_s^+.*

PROPOSITION 2.4. *A basis for Γ_s^+ is given by the monomials*

$$(2.18) \qquad\qquad \gamma = Q_{s,0}^{i_0} Q_{s,1}^{i_1} \cdots Q_{s,s-1}^{i_{s-1}}$$

for which

$$(2.19) \qquad\qquad -\infty < i_0 < \infty; \quad 0 \leq i_1, i_2, \ldots, i_{s-1}$$

and $0 \leq i_0 + \dim \gamma$.

The proofs are easy. We will show in §8 that Γ_s^+ is isomorphic to $(\Lambda^s)^*$.

Write $\Gamma^+ = \bigoplus_{s \geq 0}(\Gamma_s^+)$. We will also show in §8 how to regard Γ^+ as a quotient coalgebra of Γ. Note that Γ^+ is *not* a subcoalgebra of Γ (compute $\psi_{1,1}(Q_{2,0}^{-1}Q_{2,1}^2)$). It is nevertheless true that

$$(2.20) \qquad\qquad \psi_{s-1,1}(\Gamma_s^+) \subseteq \Gamma_{s-1}^+ \otimes \Gamma_1$$

and we will need to know this in §4. To prove (2.20) take a typical monomial $\gamma \in \Gamma_s^+$, given by (2.18). Into this expression for γ substitute (2.6) in the form $Q_{s,j} = Q_{s-1,0}Q_{s-1,j}v_s + Q_{s-1,j-1}^2$. We obtain an expansion of γ as a sum of monomials of the form $Q_{s-1,0}^{j_0} Q_{s-1,1}^{j_1} \cdots Q_{s-1,s-2}^{j_{s-2}} v_s^k$, and one verifies easily that $0 \leq j_0 + \dim \gamma - k$ in every term that appears. This proves (2.20).

As a corollary we obtain

LEMMA 2.5. *Let γ in Γ_s^+ be given; let*

$$(2.21) \qquad\qquad \gamma = \sum v_1^{j_1} \cdots v_s^{j_s}$$

be its expansion in terms of $v_1, \ldots, v_s$. Then $j_1 \geq 0$ in every monomial that appears.

This is proved by applying (2.20) repeatedly.

The reader who is interested only in the construction $\Gamma^+ M$ has already read enough invariant theory, and can proceed to §§3–5. But in order to compare $\Gamma^+ F_2$ to the dual of Λ we will need in §8 two technical lemmas (Lemmas 2.6 and 2.7 below) that relate the basis for Γ_s^+ given by Proposition 2.4 to the basis for Δ_s^+ given by Proposition 2.3. In order to state this lemma, we write I_s for the set of s-tuples of integers $(i_0, \ldots, i_{s-1})$ for which

$$(2.22) \qquad\qquad 0 \leq i_1, i_2, \ldots, i_{s-1} \quad \text{and} \quad 0 \leq i_0 + \sum_{k=0}^{s-1} (2^s - 2^k) i_k.$$

Then the set I_s labels the basis for Γ_s^+ that is given in Proposition 2.4. We write J_s for the set of s-tuples of integers $(j_1, \ldots, j_s)$ for which

$$(2.23) \qquad\qquad 0 \leq j_1 \quad \text{and} \quad j_{k-1} \leq 2 j_k \qquad (2 \leq k \leq s).$$

In §8 we will use the fact that J_s labels a basis for Λ^s. Define a function $\phi\colon I_s \to J_s$ by writing $\phi(i_0,\ldots,i_{s-1}) = (j_1, j_2,\ldots,j_s)$ where

$$(2.24) \qquad j_k = 2^{s-k}(i_0 + i_1 + \cdots + i_{k-1}) + \sum_{l=0}^{s-k-1} (2^{s-k} - 2^l)i_{k+l}.$$

Define a function $\psi\colon J_s \to I_s$ by writing $\psi(j_1,\ldots,j_s) = (i_0,\ldots,i_{s-1})$ where

$$(2.25) \qquad i_0 = j_1 - j_2 - \cdots - j_s, \qquad i_{k-1} = 2j_k - j_{k-1} \qquad (k \geqslant 2).$$

Routine checking shows

LEMMA 2.6. $\phi\colon I_s \to J_s$ is a bijection and ψ is its inverse.

To obtain further information about the basis in Proposition 2.4, let us take as a basis for Δ_s the set of monomials $\{v_1^{j_1} \cdots v_s^{j_s} \mid j_1,\ldots,j_s \in Z\}$, and order this basis antilexicographically. Simple inductive computations using (2.6) and (2.11) show

LEMMA 2.7. Let $(j_1,\ldots,j_s) = \phi(i_0,\ldots,i_{s-1})$. Then in Δ_s

$$(2.26) \qquad Q_{s,0}^{i_0}Q_{s,1}^{i_1} \cdots Q_{s,s-1}^{i_{s-1}} = v_1^{j_1} \cdots v_s^{j_s} + \textit{smaller monomials}.$$

This lemma will help us in §8 to show Γ_s^+ isomorphic to $(\Lambda^s)^*$.

3. Invariant theory and the Adem relations. In this section we show a connection between the invariant theory of GL_2 and the Adem relations. Proposition 3.1 is our central observation. It explains why invariant theory should play a role in building resolutions of modules over the Steenrod algebra.

Define an F_2-linear map $\pi\colon \Delta_2 \to A$ by

$$(3.1) \qquad\qquad\qquad \pi(v_1^p v_2^q) = \mathrm{Sq}^{p+1}\mathrm{Sq}^{q+1}.$$

Here we understand $\mathrm{Sq}^k = 0$ if $k < 0$, and $\mathrm{Sq}^0 = 1$.

PROPOSITION 3.1. $\Gamma_2 \subset \ker \pi$.

PROOF. We use the description of Γ_2 given by (2.9), and will show that $\pi(Q_{2,0}^r Q_{2,1}^s) = 0$ for all $r \in Z$ and $s \geqslant 0$ by induction on s. If $s = 0$ we note that $\mathrm{Sq}^{2r+1}\mathrm{Sq}^{r+1} = 0$ in the Steenrod algebra, so the result is proved in this case. Now we carry out the inductive step, using (3.1) to compute for any $\gamma \in \Delta_2$.

$$\pi\left(\frac{Q_{2,1}}{Q_{2,0}} \gamma \right) = \pi\left(\left(\frac{1}{v_1} + \frac{1}{v_2} \right)\gamma \right) = \alpha\pi(\gamma)$$

where $\alpha\colon A \to A$ is the derivation characterized by $\alpha(\mathrm{Sq}^k) = \mathrm{Sq}^{k-1}$ [4]. So the proposition is proved.

We remark in passing that Proposition 3.1 actually determines the Steenrod algebra. That is, one can use (3.1) to define an F_2-linear map π from Δ_2 to the tensor algebra on the formal symbols $\{\mathrm{Sq}^k \mid k > 0\}$. The Steenrod algebra results if we impose the relations $\pi(\Gamma_2) = 0$.

4. The chain complexes $\Gamma^+ M$. Let M be an arbitrary left A-module. We will now construct the chain complex $\Gamma^+ M$ having properties (1.4) and (1.5).

Define an F_2-linear map $\partial: \Delta_s \otimes M \to \Delta_{s-1} \otimes M$ by

$$(4.1) \qquad \partial\left(v_1^{j_1} \cdots v_s^{j_s} \otimes x \right) = v_1^{j_1} \cdots v_{s-1}^{j_{s-1}} \otimes \mathrm{Sq}^{j_s+1} x$$

(we understand $\mathrm{Sq}^k x = 0$ if $k < 0$). From (2.20) it is clear that $\partial(\Gamma_s^+ \otimes M) \subseteq \Gamma_{s-1}^+ \otimes M$. We define the chain complex $\Gamma^+ M$ by setting $(\Gamma^+ M)_s = \Gamma_s^+ \otimes M$ as an F_2-vector space for all $s \geq 0$, and declaring the differential to be the restriction of (4.1) to $\Gamma_s^+ \otimes M$. That the composition

$$\Gamma_s^+ \otimes M \xrightarrow{\partial} \Gamma_{s-1}^+ \otimes M \xrightarrow{\partial} \Gamma_{s-2}^+ \otimes M$$

is zero follows from Proposition 2.1 (with $q = 2$) and Proposition 3.1.

We regard the association $M \to \Gamma^+ M$ as a functor from A-modules to chain complexes over F_2, and prove at once the following three propositions.

PROPOSITION 4.1. *There is a natural isomorphism $H_0(\Gamma^+ M) = F_2 \otimes_A M$.*

PROPOSITION 4.2. *If $0 \to M \to N \to P \to 0$ is a short exact sequence of A-modules then $0 \to \Gamma^+ M \to \Gamma^+ N \to \Gamma^+ P \to 0$ is a short exact sequence of chain complexes.*

PROPOSITION 4.3. *To any short exact sequence $0 \to M \to N \to P \to 0$ of A-modules is naturally associated a long exact sequence in homology:*

$$\cdots \to H_s(\Gamma^+ N) \to H_s(\Gamma^+ P) \to H_{s-1}(\Gamma^+ M) \to \cdots \to H_0(\Gamma^+ P) \to 0.$$

Finally we assert

PROPOSITION 4.4. $H_s(\Gamma^+ A) = 0$ *if $s > 0$.*

PROOF. Our proof is in fact dual to the argument in §7.4 of [1], but we include it to make our paper selfcontained. If $\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_p}$ is an "admissible" monomial in A, write $l(I) = p$ for its length. Define an increasing filtration on $\Gamma^+ A$ by setting

$$F_p(\Gamma^+ A)_s = \Gamma_s^+ \otimes \mathrm{Span}\left\{ \mathrm{Sq}^I \mid I \text{ admissible}; \, l(I) \leq p - s \right\}.$$

Then $F_p(\Gamma^+ A)$ is a subcomplex of $\Gamma^+ A$ for each $p \geq 0$. Write $E^0 \Gamma^+ A$ for the associated graded object: for each $p \geq 0$, $E_p^0 \Gamma^+ A = F_p \Gamma^+ A / F_{p-1} \Gamma^+ A$ is a chain complex. Now, a monomial $y = Q_{s,0}^{i_0} \cdots Q_{s,s-1}^{i_{s-1}} \otimes \mathrm{Sq}^{k_1} \cdots \mathrm{Sq}^{k_p}$, in $F_p(\Gamma^+ A)_s / F_{p-1}(\Gamma^+ A)_s$ will be called admissible if both $\mathrm{Sq}^{k_1} \cdots \mathrm{Sq}^{k_p}$, is an admissible monomial in A, and if $j_s > 2k_1 - 2$. Here we are writing $(j_1, \ldots, j_s) = \phi(i_0, \ldots, i_{s-1})$ as in Lemma 2.6; in particular $j_s = i_0 + i_1 + \cdots + i_{s-1}$. Now define an increasing filtration on $E^0 \Gamma^+ A$ by setting

$$\left(G_q E_0 \Gamma^+ A \right)_s = \begin{cases} E_0 \Gamma^+ A, & s < q, \\ \mathrm{Span}\left\{ y = Q_{s,0}^{i_0} \cdots Q_{s,s-1}^{i_{s-1}} \otimes \mathrm{Sq}^I \mid y \text{ admissible} \right\}, & s = q, \\ 0, & s > q, \end{cases}$$

for all $q \geq 0$. Then $G_q E_0 \Gamma^+ A$ is a subcomplex of $E^0 \Gamma^+ A$. Using Proposition 2.4 and Lemma 2.6 we see easily that

$$H_s \left(\frac{G_q E_0 \Gamma^+ A}{G_{q-1} E_0 \Gamma^+ A} \right) = \begin{cases} F_2 & \text{if } s = q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.4 follows immediately.

THEOREM 4.5. *There is a natural isomorphism $H_s(\Gamma^+ M) \simeq \operatorname{Tor}_s^A(F_2, M)$.*

PROOF. The properties of the functors $M \to H_s(\Gamma^+ M)$ ($s = 0, 1, 2, \ldots$) given by Propositions 4.1, 4.3, and 4.4 are also properties of the functors $M \to \operatorname{Tor}_s^A(F_2, M)$; and in fact, the latter functors are characterized by them.

This is Theorem 1.1 of the introduction.

5. Action of the Steenrod algebra on $(\Gamma^+ M)_s$. Our object in this section is to prove

THEOREM 5.1. *Let M be a graded A-module. For each fixed $s \geq 0$, $(\Gamma^+ M)_s$ has a natural left A-action, in such a way that the differential $\partial: (\Gamma^+ M)_s \to (\Gamma^+ M)_{s-1}$ is A-linear.*

In order to show this, we recall the main results of [8]. There we constructed a functor R_+ from the category of graded A-modules to itself. In the present context Δ seems a more appropriate name for this functor; we will use it here. The main properties of this functor are:

(i) ΔM is isomorphic to $\Delta_1 \otimes M$ as a graded F_2-vector space (*not* in general as an A-module). Here $\Delta_1 = F_2[v^{\pm 1}]$ is the algebra of finite Laurent series on a generator v of dimension one, as in (2.12). The A-action on ΔM is given by

$$(5.1) \qquad \operatorname{Sq}^a(v^b \otimes x) = \sum_j \binom{b-j}{a-2j} v^{a+b-j} \otimes \operatorname{Sq}^j x$$

with the binomial coefficient interpreted as in §2 of [8]. If $g: M \to N$ is an A-linear map, then $\Delta g: \Delta M \to \Delta N$ is given by $(\Delta g)(v^k \otimes x) = v^k \otimes g(x)$.

(ii) If $d: \Delta M \to M$ is the "evaluation map" given by $d(v^k \otimes x) = \operatorname{Sq}^{k+1} x$, then d is A-linear.

These results are contained in Theorem 3.4 of [8].

We can now explain our strategy for proving Theorem 5.1. Write $\Delta_s M$ for the result of applying the functor Δ to M a total of s-times. (In [8], $\Delta_s M$ was written $R_+^s M$.) Then as a graded vector space, $\Delta_s M$ is obtained by tensoring M with an algebra of finite Laurent series on s generators, each of dimension one. We will identify this algebra of Laurent series with the algebra Δ_s of (2.8) and (2.12), writing

$$(5.2) \qquad \Delta_s M = \Delta_s \otimes M = F_2[v_1^{\pm 1}, \ldots, v_s^{\pm 1}] \otimes M$$

as graded vector spaces. Here the identification of left-hand side with right-hand side is defined inductively by the following rule. If $y \in \Delta_{s-1} M$ has been identified with $v_1^{j_1} \cdots v_{s-1}^{j_{s-1}} \otimes x$, then $v^k \otimes y$ in $\Delta(\Delta_{s-1} M) = \Delta_s M$ is to be identified with $v_1^k v_2^{j_1} \cdots v_s^{j_{s-1}} \otimes x$ in $\Delta_s \otimes M$. (5.2) is an isomorphism of F_2 vector spaces only. It is not meant to imply anything about $\Delta_s M$ as an A-module.

We define F_2-subspaces $\Delta_s^+ M$, $\Gamma_s M$, and $\Gamma_s^+ M$ of $\Delta_s M$, by using the identification (5.2) and writing

$$(5.3) \qquad \Delta_s^+ M = \Delta_s^+ \otimes M, \quad \Gamma_s M = \Gamma_s \otimes M, \quad \Gamma_s^+ M = \Gamma_s^+ \otimes M.$$

We will show

THEOREM 5.2. *Let M be a graded A-module. Then $\Delta_s^+ M$, $\Gamma_s M$, and $\Gamma_s^+ M$ are all A-submodules of $\Delta_s M$.*

Before stating the proof of Theorem 5.2 we observe that it implies Theorem 5.1. In fact $(\Gamma^+ M)_s$ was defined in §4 to be the vector space $\Gamma_s^+ \otimes M$, so Theorem 5.2 gives us a natural A-action on $(\Gamma^+ M)_s$. We demonstrate the A-linearity of $\partial: (\Gamma^+ M)_s \to (\Gamma^+ M)_{s-1}$ by observing that with the identification (5.2), $\partial: \Delta_s \otimes M \to \Delta_{s-1} \otimes M$ of (4.1) is just $\Delta_{s-1}d: \Delta_s M \to \Delta_{s-1}M$, where $d: \Delta M \to M$ is the A-linear evaluation map of (ii) above. So to prove Theorem 5.1, we need only prove Theorem 5.2.

It is helpful to begin with the case in which the A-module M is unstable. In this case we define a subobject $RM \subseteq \Delta M$:

$$(5.4) \qquad RM = \text{Span}\{v^b \otimes x \,|\, b \geq \dim x\}.$$

We use (5.1) to check that RM is an A-submodule of ΔM, and that RM is unstable if M is. (The functor R was introduced in [10].)

Write $M \underline{\otimes} N$ for the tensor product of two A-modules with *diagonal A-action*. In order to study the A-action upon RM we recall another result from [8]: an idea of Haynes Miller.

(iii) Suppose M an unstable A-module. Then $f: RM \to P_1 \underline{\otimes} M$ given by

$$(5.5) \qquad f(v^b \otimes x) = \sum_{j=0}^{b} \left(t^{b-j} \otimes \text{Sq}^j x \right)$$

is an A-linear monomorphism.

This comes from §3 of [8]. There it was assumed that M was bounded above, but is clear that f can also be defined if M is unstable. An alternate expression for f is (3.3) of [8], valid when M is unstable:

$$(5.6) \qquad f(v^b \otimes x) = \text{Sq}^{b+1}(t^{-1} \otimes x).$$

The right-hand side of (5.6) refers to the diagonal action of A upon $\Delta_1 \underline{\otimes} M$, and the action of A upon Δ_1 is as in (2.1) of [8].

Now write R_s for the s-fold iterate of R. Then for any unstable A-module M, $R_s M$ is naturally an A-submodule of $\Delta_s M$. For integers $s \geq 1$ and unstable M, define by induction on s an A-linear monomorphism

$$(5.7) \qquad f_s: R_s M \to P_s \underline{\otimes} M$$

by setting $f_1 = f$, and declaring commutative the diagram:

$$(5.8) \quad
\begin{array}{ccc}
R_s M = R R_{s-1} M & \overset{Rf_{s-1}}{\to} & R\left(P_{s-1} \underline{\otimes} M \right) \\
f_s \downarrow & & \downarrow f_1 \\
P_s \underline{\otimes} M & \underset{\cong}{\to} & P_1 \underline{\otimes} \left(P_{s-1} \underline{\otimes} M \right)
\end{array}$$

(Of course one must be aware that $P_{s-1} \otimes M$ is unstable if M is.) We obtain an equivalent definition of f_s by writing $f_s = (P_1 \otimes f_{s-1}) \circ f_1$. As a consequence of this equivalence we have commutativity of the following diagram for all $i, 0 \leqslant i \leqslant s - 2$:

$$
(5.9) \quad
\begin{array}{ccccc}
R^s M & \xrightarrow{\sim} & R^i R^2 R^{s-i-2} M & \xrightarrow{R^i f_2} & R^i \left(P_2 \otimes R^{s-i-2} M \right) \\
f_s \downarrow & & & & \downarrow R^i \left(P_2 \otimes f_{s-i-2} \right) \\
P_s \otimes M & \xrightarrow[\sim]{} & P_i \otimes P_2 \otimes P_{s-i-2} \otimes M & \xleftarrow{f_i} & R^i \left(P_2 \otimes P_{s-i-2} \otimes M \right)
\end{array}
$$

When we make the identification (5.2), elements of the submodule $R_s M \subset \Delta_s M$ are represented by elements of the submodule $P_s \otimes M \subset \Delta_s \otimes M$. This observation helps us to understand the following lemma, which gives another property of $f_s: R_s M \to P_s \otimes M$.

LEMMA 5.3. *Let $\gamma \otimes x$ represent an element of $R_s M$ under the identification (5.2), where $\gamma \in P_s$, and $x \in M$ is homogeneous. Then $f_s(\gamma \otimes x) = \gamma \otimes x + \Sigma_i \gamma_i \otimes x_i$ where $\dim x_i > \dim x$ for all x_i appearing in the sum.*

This follows directly from the definition of f_s.
Our first result en route to Theorem 5.2 is

LEMMA 5.4. *Let M be an unstable A-module. Then the image of $f_2: R_2 M \to P_2 \otimes M$ is contained in $P_2^{T_2} \otimes M$.*

PROOF. Using (5.5), (5.6), and (5.8) we find

$$
(5.10) \quad f_2 \left(v_1^a v_2^b \otimes x \right) = \mathrm{Sq}^{a+1} \sum_{j=0}^{b} t_1^{-1} t_2^{b-j} \otimes \mathrm{Sq}^j x.
$$

But since $v_1^a v_2^b \otimes x$ lies in $R_2 M$ we have $a \geqslant b + \dim x$. Then it follows from the instability of $P_2 \otimes M$ that the right-hand side of (5.10) is unaffected by the substitution $t_1 \to t_1$, $t_2 \to t_1 + t_2$. This completes the proof.

LEMMA 5.5. *Let M be unstable. Then f_2 carries $R_2 M \cap \Gamma_2 M$ into $P_2^{GL_2} \otimes M$.*

PROOF. $R_2 M \cap \Gamma_2 M$ is spanned by elements of the form $\gamma \otimes x$, where $x \in M$ and $\gamma = \Sigma_i v_1^{a_i} v_2^{b_i}$ lies in Γ_2. Then by Proposition 3.1, $\Sigma_i \mathrm{Sq}^{a_i+1} \mathrm{Sq}^{b_i+1} = 0$. Then in $P_2 \otimes M$ we have

$$
(5.11) \quad
\begin{aligned}
0 &= \sum_i \mathrm{Sq}^{a_i+1} \mathrm{Sq}^{b_i+1} \left(t_1^{-1} t_2^{-1} \otimes x \right) \\
&= \sum_i \mathrm{Sq}^{a_i+1} \left(t_1^{-1} \mathrm{Sq}^{b_i+1} \left(t_2^{-1} \otimes x \right) \right) \\
&\quad + \sum_i \mathrm{Sq}^{a_i+1} \sum_{j=0}^{b_i} t_1^{b_i-j} \mathrm{Sq}^j \left(t_2^{-1} \otimes x \right).
\end{aligned}
$$

Using (5.6) we see that (5.11) implies

$$
(5.12)
$$

$$
f_2(\gamma \otimes x) = \sum_i \mathrm{Sq}^{a_i+1} \left(t_1^{-1} \mathrm{Sq}^{b_i+1} \left(t_2^{-1} \otimes x \right) \right) = \sum_i \mathrm{Sq}^{a_i+1} \sum_{j=0}^{b_i} t_1^{b_i-j} \mathrm{Sq}^j \left(t_2^{-1} \otimes x \right).
$$

But if $\gamma \otimes x$ lies in $R_2 M$ then $a_i \geq b_i + \dim x$ for all i, so using the instability of $P_2 \underline{\otimes} M$ we obtain from (5.12),

$$(5.13) \qquad f_2(\gamma \otimes x) = \sum_i \mathrm{Sq}^{a_i+1} \sum_{j=0}^{b_i} t_1^{b_i-j} t_2^{-1} \otimes \mathrm{Sq}^j x.$$

Comparing this expression for $f_2(\gamma \otimes x)$ with the one coming from (5.10), we find that $f_2(\gamma \otimes x)$ is unchanged by the substitution $t_1 \to t_2$, $t_2 \to t_1$. Appeal to Lemma 5.4 completes the proof.

Now we generalize Lemma 5.5.

LEMMA 5.6. *Let M be unstable. Then f_s carries $R_s M \cap \Gamma_s M$ into $P_s^{GL_s} \underline{\otimes} M$.*

PROOF. Let i be any integer, $0 \leq i \leq s - 2$. As vector spaces we can identify $\Delta_s M \cong \Delta_s \otimes M \cong \Delta_i \otimes \Delta_2 \otimes \Delta_{s-i-2} \otimes M$, where we are using the maps $\psi_{p,q}$ of (2.13) to define the isomorphism

$$\Delta_s \overset{\psi_{i,s-i}}{\to} \Delta_i \otimes \Delta_{s-i} \overset{\Delta_i \otimes \psi_{2,s-i-2}}{\to} \Delta_i \otimes \Delta_2 \otimes \Delta_{s-i-2}.$$

Then it follows from Proposition 2.1 that any $y \in \Gamma_s M$ can be written as a sum of elements of the form $\gamma_1 \otimes \gamma_2 \otimes \gamma_3 \otimes x$, with $\gamma_1 \in \Delta_i$, $\gamma_2 \in \Gamma_2$, $\gamma_3 \in \Delta_{s-i-2}$. Then it follows from (5.9) and Lemma 5.5 that $f_s(y)$ lies in $P_i \otimes P_2^{GL_2} \otimes P_{s-i-2} \otimes M$. But this is true for any i with $0 \leq i \leq s - 2$ so our lemma is proved.

Lemma 5.6 can be used to prove its own converse.

LEMMA 5.7. *Let M be unstable. Then $f_s^{-1}(P_s^{GL_s} \underline{\otimes} M) \subseteq R_s M \cap \Gamma_s M$.*

PROOF. This follows easily from Lemmas 5.3 and 5.6.
We now can prove a special case of Theorem 5.2.

PROPOSITION 5.8. *Let M be unstable. Then $R_s M \cap \Gamma_s M$ is an A-submodule of $R_s M$.*

PROOF. This follows at once from Lemmas 5.6 and 5.7, and the obvious fact that $P_s^{GL_s} \underline{\otimes} M$ is an A-submodule of $P_s \underline{\otimes} M$.

We next aim to generalize Proposition 5.8, by showing that if M is unstable then $\Gamma_s M$ is an A-submodule of $\Delta_s M$. To this end we continue to recall from [8] the properties of the functor Δ.

(iv) The natural pairing $\xi = \xi(M, N): \Delta M \underline{\otimes} \Delta N \to \Delta(M \underline{\otimes} N)$ given by

$$(5.14) \qquad \xi\big((v^a \otimes x) \otimes (v^b \otimes y)\big) = v^{a+b} \otimes (x \otimes y)$$

is an A-linear map. In particular if M is an A-algebra with product $\mu: M \underline{\otimes} M \to M$ (as in [8, §2]), then ΔM is also an A-algebra if we define the product by

$$\Delta M \underline{\otimes} \Delta M \overset{\xi}{\to} \Delta(M \underline{\otimes} M) \overset{\Delta\mu}{\to} \Delta M.$$

(v) If $M = F_2$, the identification (5.2) of $\Delta_s F_2$ with Δ_s is an isomorphism of A-algebras.

Property (iv) above is Proposition 7.1 of [8]. (v) is obtained by combining Propositions 9.1 and 9.2 of [8]. Of course, the A-action on Δ_s that we refer to here is obtained by applying the process of localizing an A-algebra, due to Wilkerson

[**11**, Proposition 2.1], to the A-algebra $P_s^{T_s}$. Alternately, one can start with the action of A upon P_s that comes from identifying P_s with the cohomology of a product of infinite projective spaces, pass by Wilkerson's procedure to an action of A upon the localization $\Phi_s = (P_s)_{S(s)}$, and then restrict this action to Δ_s. These matters are all explained in §§2, 8 and 9 of [**8**].

For any A-module M, define an action

$$(5.15) \qquad \Delta_s \underline{\otimes} \Delta_s M \to \Delta_s M$$

by using the identification (5.2) and writing $\gamma(\gamma' \otimes x) = \gamma\gamma' \otimes x$.

PROPOSITION 5.9. *With diagonal A-action as indicated on the left-hand side, (5.15) is an A-linear map.*

PROOF. (5.15) can be obtained by repeated use of the pairing (5.14)

Now we can extend Proposition 5.8.

PROPOSITION 5.10. *Let M be unstable. Then $\Gamma_s M$ is an A-submodule of $\Delta_s M$.*

PROOF. Suppose we wish to show that $\mathrm{Sq}^k(\gamma \otimes x)$ lies in $\Gamma_s M$, where $\gamma \in \Gamma_s$ and we are using (5.2) to label elements of $\Delta_s M$. But it is easy to see that we can choose elements $\alpha, \beta \in \Gamma_s$ such that $\gamma = \alpha\beta$ and $\beta \otimes x$ lies in $R_s M \cap \Gamma_s M$. Then by Proposition 5.9 we have

$$(5.16) \qquad \mathrm{Sq}^k(\gamma \otimes x) = \sum_{i+j=k} \mathrm{Sq}^i \alpha \cdot \mathrm{Sq}^j(\beta \otimes x).$$

By Proposition 5.8, each $\mathrm{Sq}^j(\beta \otimes x)$ lies in $\Gamma_s M$. Also, reference to the invariant-theoretic definitions of Γ_s and Δ_s shows that Γ_s is an A-subalgebra of Δ_s, so each $\mathrm{Sq}^i \alpha$ lies in Γ_s. Hence the right-hand side of (5.16) lies in $\Gamma_s M$ and we are done.

We can now easily remove from Proposition 5.10 the assumption that M is unstable. First we have

PROPOSITION 5.11. *Let M be a graded A-module, bounded above. Then $\Gamma_s M$ is an A-submodule of $\Delta_s M$.*

PROOF. Suppose we wish to show that $\mathrm{Sq}^k(\gamma \otimes x)$ lies in $\Gamma_s M$, where $\gamma \in \Gamma_s$. Let $M' \subseteq M$ be the A-submodule consisting of all elements y for which $\dim x \leqslant \dim y$. It is enough to show that $\mathrm{Sq}^k(\gamma \otimes x)$ lies in $\Gamma_s M'$. But the r-fold suspension $\Sigma^r M'$ is unstable for r sufficiently large (M is bounded above) so our result follows from the fact that the functors Σ and Δ commute.

PROPOSITION 5.12. *Let M be an arbitrary graded A-module. Then $\Gamma_s M$ is an A-submodule of $\Delta_s M$.*

PROOF. This follows easily from the case in which M is bounded above.

We continue toward a proof of Theorem 5.2 by showing

PROPOSITION 5.13. *Let M be a graded A-module. Then $\Delta_s^+ M$ is an A-submodule of $\Delta_s M$.*

PROOF. By Proposition 2.3, it will suffice to show that if the monomial $v_1^{j_1} \cdots v_s^{j_s}$ lies in Δ_s^+, then for any $x \in M$ the element $\mathrm{Sq}^k(v_1^{j_1} \cdots v_s^{j_s} \otimes x)$ lies in $\Delta_s^+ M$. But

repeated use of (5.1) shows that $\mathrm{Sq}^k(v_1^{j_1} \cdots v_s^{j_s} \otimes x)$ can be written as a sum of terms of the form $v_1^{l_1} \cdots v_s^{l_s} \otimes y$, where

$$(5.17) \qquad\qquad\qquad j_1 \leqslant l_1, \ldots, j_s \leqslant l_s.$$

Now each v_i lies in Δ_s^+ (apparent from (2.2), (2.10) and (2.15)), and since Δ_s^+ is a subalgebra of Δ_s, it follows from (5.17) that $v_1^{l_1} \cdots v_s^{l_s}$ lies in Δ_s^+. The proof is complete.

PROPOSITION 5.14. *Let M be a graded A-module. Then $\Gamma_s^+ M$ is an A-submodule of $\Delta_s M$.*

PROOF. Since $\Gamma_s^+ = \Gamma_s \cap \Delta_s^+$, our result follows from Propositions 5.12 and 5.13.

Combining Propositions 5.12–5.14, we find that we have proved Theorem 5.2. As we remarked after the statement of that theorem, Theorem 5.1 follows immediately.

Since we have now shown that the chain complex $\Gamma^+ M$ has a natural A-action in each homological degree, it is natural to ask about the induced A-action on each $\mathrm{Tor}_s^A(F_2, M) = H_s(\Gamma^+ M)$.

THEOREM 5.15. *The action of A upon $\Gamma^+ M$ induces the trivial action of A upon $\mathrm{Tor}_s^A(F_2, M)$, for each $s \geqslant 0$.*

PROOF. Induct on s. For the case $s = 0$ observe that the chain complex $\Gamma^+ M$ in homological dimensions 0 and 1 looks like

$$\mathrm{Span}\{v_1^k \,|\, k \geqslant 0\} \otimes M \overset{\partial}{\to} M$$

where $\partial(v_1^k \otimes x) = \mathrm{Sq}^{k+1}x$. The cases $s > 0$ do *not* follow in such a straightforward way from the structure of $\Gamma^+ M$. But we use the inductive assumption: suppose the result proved for the case $s - 1$, for some $s \geqslant 1$ and all A-modules M. Let $\pi: F \to M$ be a mapping from a free A-module onto M, and let K be the kernel. Then $0 \to \Gamma^+ K \to \Gamma^+ F \to \Gamma^+ M \to 0$ is a short exact sequence of chain complexes of A-modules, of which the middle complex is acyclic in positive homological dimensions. So the triviality of the A-action on $H_s(\Gamma^+ M)$ follows from the triviality of the A-action on $H_{s-1}(\Gamma^+ K)$.

Theorems 5.1 and 5.15 together give Theorem 1.3 of the introduction.

6. An extension of the lambda algebra. The next three sections are devoted to comparing the chain complex $\Gamma^+ F_2$ with the dual of the lambda algebra. We will give $\Gamma^+ F_2$ the structure of a differential coalgebra, and prove Theorem 1.2. We begin by defining lambda by embedding it in a somewhat larger differential algebra, which we will call "theta". This algebra was described to me by Bousfield and by Priddy, and was known to the authors of [1].

Let L be the graded F_2-vector space with basis consisting of the symbols $\{\lambda_k \,|\, k$ integral, $k \geqslant -1\}$, where $\deg \lambda_k = k$. Let $\mathrm{Tens}\, L$ be the free associative algebra generated by L. Then $\mathrm{Tens}\, L$ is a bigraded algebra if we write bideg $\lambda_k = (1, k)$. We will usually supress the second of these two degrees ("interval degree") and record

only the first ("homological degree"). In $(\text{Tens } L)^2 = L \otimes L$ we define a family of homogeneous elements

$$(6.1) \qquad \lambda(p, q) = \sum_{j \geq 0} \binom{p}{j} \lambda_{2q+j-1} \lambda_{p+q-j-1} \qquad (p, q \geq 0)$$

and then define Θ to be the bigraded algebra obtained from Tens L by imposing the relations $\lambda(p, q) = 0$ ($p, q \geq 0$). These relations are of two kinds. Those involving λ_{-1} are precisely the statements $\lambda(p, 0) = 0$:

$$\lambda_{-1}\lambda_{-1} = 0,$$

$$(6.2) \qquad \lambda_{-1}\lambda_{p-1} + \sum_{j=1}^{p-1} \binom{p}{j} \lambda_{j-1}\lambda_{p-j-1} + \lambda_{p-1}\lambda_{-1} = 0 \qquad (p > 0)$$

but only the generators $\{\lambda_k \mid k \geq 0\}$ appear in $\lambda(p, q)$ if $q > 0$.

Now we let Λ be the subalgebra of Θ that is generated by the symbols $\{\lambda_k \mid k \geq 0\}$. Then Λ is defined by the relations $\lambda(p, q) = 0$ for which $p \geq 0$ and $q > 0$. This definition is the same as that originally given in [1], but observe that we are writing the product in the order opposite to that used in [1]. We note, as in [2, p. 459], that for each $s \geq 1$, a basis for Λ^s is given by the set of "admissibles":

$$(6.3) \qquad \{\lambda_{j_1}\lambda_{j_2} \cdots \lambda_{j_s} \mid 0 \leq j_1, \ j_1 \leq 2j_2, \ldots, j_{s-1} \leq 2j_s\}$$

while Λ^0 is spanned by the unit.

Define a homomorphism $d: \Theta \to \Theta$ by

$$(6.4) \qquad dx = \lambda_{-1}x + x\lambda_{-1}.$$

This definition was shown me by Bousfield and Priddy. The map d is clearly a derivation, and since $\lambda_{-1}\lambda_{-1} = 0$ we have $dd = 0$. (6.2) also implies that

$$(6.5) \qquad d\lambda_{p-1} = \sum_{j=1}^{p-1} \binom{p}{j} \lambda_{j-1}\lambda_{p-j-1} \qquad (p > 0)$$

so that Λ is a subdifferential algebra of Θ. This completes our description of the differential algebras Θ and Λ.

We close this section by defining a certain derivation $\chi: \Theta \to \Theta$; it will be useful in §7 when we compare Γ^+ to Θ^*. In fact there is a unique derivation χ: Tens $L \to$ Tens L that satisfies

$$(6.6) \qquad \chi(\lambda_k) = \lambda_{k+1} \qquad (k \geq -1).$$

An easy computation shows

$$(6.7) \qquad \chi\lambda(p, q) = \lambda(p + 1, q)$$

so χ passes to a derivation on Θ satisfying (6.6).

7. Invariants of GL_s and the dual of Θ. Our object in this section is to construct a map of coalgebras $k: \Gamma \to \Theta^*$.

If V is a graded (or bigraded) vector space, write V^* for its graded (bigraded) dual: $(V^*)_m = (V_m)^*$ (or $(V^*)_{p,q} = (V_{p,q})^*$). Then the equations $(V^*)^* = V$ and $(V \otimes W)^* = V^* \otimes W^*$ are valid as long as V and W are finite dimensional in each

degree (or bidegree). In particular, we may regard $(\mathrm{Tens}\, L)^*$, Θ^*, and Λ^* as coalgebras.

Define the degree-preserving F_2-linear $k_s\colon \Delta_s \to (L^{\otimes s})^*$ by

$$(7.1) \qquad \left\langle k_s\big(v_1^{i_1}\cdots v_s^{i_s}\big),\, \lambda_{j_1}\otimes\cdots\otimes\lambda_{j_s}\right\rangle = \delta_{i_1,j_1}\cdot\cdots\cdot\delta_{i_s,j_s}$$

for each $s \geq 1$, where $\delta_{i,j}$ is the Kronecker delta. Let $k_0\colon \Delta_0 \to (L^{\otimes 0})^*$ be the identity on F_2. With $\psi_{p,q}\colon \Delta_s \to \Delta_p \otimes \Delta_q$ as in (2.13) we have clearly

$$(7.2) \qquad \left\langle k_s(\gamma),\, \alpha\beta\right\rangle = \left\langle (k_p \otimes k_q)\psi_{p,q}(\gamma),\, \alpha\otimes\beta\right\rangle$$

for $\alpha \in L^{\otimes p}$, $\beta \in L^{\otimes q}$, $\gamma \in \Delta_s$. (We are identifying $(L^{\otimes p} \otimes L^{\otimes q})^*$ with $(L^{\otimes p})^* \otimes (L^{\otimes q})^*$ in (7.2).) That is, if the maps k_s for $s \geq 0$ are assembled to form $k\colon \Delta \to (\mathrm{Tens}\, L)^*$ then

PROPOSITION 7.1. $k\colon \Delta \to (\mathrm{Tens}\, L)^*$ *is a morphism of graded coalgebras.*

We now aim to show that the restriction $k_s \,|\, \Gamma_s$ is actually a map from Γ_s to $(\Theta^s)^*$; that is, if γ lies in Γ_s then the functional $k_s(\gamma)$ annihilates the relations (6.1) by which Θ is defined. We begin by proving this in the case $s = 2$.

LEMMA 7.2. *For any* $\gamma \in \Gamma_2$ *and any integers* $p, q \geq 0$ *we have*

$$(7.3) \qquad\qquad \left\langle k_2(\gamma),\, \lambda(p,q)\right\rangle = 0.$$

PROOF. It suffices to prove this for the case $\gamma = Q_{2,0}^a Q_{2,1}^b$, where a, b are integers and $0 \leq b$. In the case $b = 0$ we note that the relation $\left\langle k_2(Q_{2,0}^a),\, \lambda(p,q)\right\rangle = 0$ follows easily from (6.1), (7.1), and the fact that $\binom{3l+1}{2l+1} = 0$ modulo 2 for all values of the integer l. Now we induct on b. Suppose (7.3) proved for $\gamma = Q_{2,0}^a Q_{2,1}^b$ for some fixed $b \geq 0$ and all values of a, p, and q. Define $\rho\colon \Delta_2 \to \Delta_2$ by writing

$$(7.4) \qquad\qquad \rho(\delta) = \frac{Q_{2,1}}{Q_{2,0}}\delta = \left(\frac{1}{v_1} + \frac{1}{v_2}\right)\delta$$

for each $\delta \in \Delta_2$. Clearly the diagram

$$(7.5) \qquad
\begin{array}{ccc}
\Delta_2 & \overset{k_2}{\to} & (L \otimes L)^* \\
\rho\downarrow & & \downarrow \chi^* \\
\Delta_2 & \underset{k_2}{\to} & (L \otimes L)^*
\end{array}
$$

commutes, where χ is the derivation of $\mathrm{Tens}\, L$ defined in the previous section. Now if we let $\gamma = Q_{2,0}^a Q_{2,1}^b$ we have

$$\left\langle k_2\big(Q_{2,0}^{a-1} Q_{2,1}^{b+1}\big),\, \lambda(p,q)\right\rangle = \left\langle k_2\rho(\gamma),\, \lambda(p,q)\right\rangle = \left\langle \chi^* k_2(\gamma),\, \lambda(p,q)\right\rangle$$
$$= \left\langle k_2(\gamma),\, \chi\lambda(p,q)\right\rangle = \left\langle k_2(\gamma),\, \lambda(p+1,q)\right\rangle = 0$$

where we have used (6.7) and the inductive assumption. So our proof of (7.3) is complete.

Now Lemma 7.2 generalizes to

PROPOSITION 7.3. *Let* $\beta \in L^{\otimes s}$ *lie in the two-sided ideal of* $\mathrm{Tens}\, L$ *that is generated by the elements* $\lambda(p,q)$. *Then* $\left\langle k_s(\gamma),\, \beta\right\rangle = 0$ *for any* $\gamma \in \Gamma_s$.

PROOF. We may assume $\beta = \alpha_1\alpha_2\alpha_3$ where $\alpha_1 \in L^{\otimes r}$, $\alpha_2 = \lambda(p, q) \in L^{\otimes 2}$, and $\alpha_3 \in L^{\otimes s - r - 2}$ for some r. Our result then follows from (7.2), Proposition 2.1, and Lemma 7.2.

It follows from Proposition 7.3 that the restriction of $k_s : \Delta_s \to (L^{\otimes s})^*$ to Γ_s gives an F_2-linear map

$$(7.6) \qquad\qquad k_s : \Gamma_s \to (\Theta^s)^*.$$

If we assemble the various k_s ($s \geq 0$) to form $k : \Gamma \to \Theta^*$, then from Propositions 2.1 and 7.1 we have

PROPOSITION 7.4. $k : \Gamma \to \Theta^*$ is a morphism of graded coalgebras.

8. Invariants of GL_s and the dual of Λ. In this section we will use the coalgebra map $k : \Gamma \to \Theta^*$ to show how Γ^+ can be regarded as a quotient coalgebra of Γ. We will also prove the isomorphism of differential coalgebras $(\Gamma^+ = \Gamma^+ F_2) \to \Lambda^*$ that we promised in Theorem 1.2.

Let $\Gamma_s^+ \subseteq \Gamma_s$ be the subalgebra defined by (2.17). Restricting k_s of (7.6) to this subalgebra and then composing with the natural projection $(\Theta^s)^* \to (\Lambda^s)^*$ we get an F_2-linear map

$$(8.1) \qquad\qquad l_s : \Gamma_s^+ \to (\Lambda^s)^*.$$

PROPOSITION 8.1. l_s is an isomorphism for each $s \geq 0$.

PROOF. Let us take as a basis for $(\Lambda^s)^*$ the one dual to the basis of admissibles (6.3), writing a typical element $(\lambda_{j_1} \cdots \lambda_{j_s})^*$, where $(j_1, \ldots, j_s)$ lies in the set J_s defined by (2.23). We order this basis antilexicographically. Now take as a basis for Γ_s^+ the set of monomials given in Proposition 2.4. This basis is labeled by the sequences $(i_0, \ldots, i_{s-1})$ which lie in the set I_s, defined by (2.22). But from (7.1) and Lemma 2.7 we find that

$$l_s\big(Q_{s,0}^{i_0} \cdots Q_{s,s-1}^{i_{s-1}}\big) = \big(\lambda_{j_1} \cdots \lambda_{j_s}\big)^* + \text{smaller basis elements}$$

where $(i_0, \ldots, i_{s-1})$ is any element of I_s, and $\phi(i_0, \ldots, i_{s-1}) = (j_1, \ldots, j_s)$. Our result now follows from Lemma 2.6.

By identifying Γ_s^+ with $(\Gamma^+ F_2)_s$ we obtain a differential $\partial : \Gamma_s^+ \to \Gamma_{s-1}^+$ defined as in (4.1). Clearly this differential is obtained by restricting to Γ_s^+ the map $\partial : \Delta_s \to \Delta_{s-1}$ given by

$$(8.2) \qquad\qquad \partial\big(v_1^{j_1} \cdots v_s^{j_s}\big) = \begin{cases} v_1^{j_1} \cdots v_{s-1}^{j_{s-1}} & \text{if } j_s = -1, \\ 0 & \text{if } j_s \neq -1. \end{cases}$$

On the other hand the differential $d. \Lambda^{s-1} \to \Lambda^s$, defined either by (6.4) or (6.5), dualizes to $d^* : (\Lambda^s)^* \to (\Lambda^{s-1})^*$, and we augment Proposition 8.1 by proving

PROPOSITION 8.2. For each $s \geq 1$ the following diagram commutes:

$$(8.3) \qquad\qquad \begin{array}{ccc} \Gamma_s^+ & \overset{l_s}{\to} & (\Lambda^s)^* \\ \partial \downarrow & & \downarrow d^* \\ \Gamma_{s-1}^+ & \underset{l_{s-1}}{\to} & (\Lambda^{s-1})^* \end{array}$$

PROOF. Let $\gamma \in \Gamma_s^+$ be given, and consider the expansion (2.21). We will show

$$(8.4) \qquad l_{s-1}\partial\left(v_1^{j_1}\cdots v_s^{j_s}\right) = d^*l_s\left(v_1^{j_1}\cdots v_s^{j_s}\right)$$

for each monomial on the right of (2.21). Consider first the case $j_s = -1$. Evaluating both sides of (8.4) on a test element of Λ^{s-1} we find with the aid of (7.1),

$$(8.5) \qquad \left\langle l_{s-1}\partial\left(v_1^{j_1}\cdots v_s^{-1}\right), \lambda_{k_1}\cdots\lambda_{k_{s-1}}\right\rangle = \delta_{j_1,k_1}\cdots\delta_{j_{s-1},k_{s-1}}$$

and with the aid of (6.4),

$$(8.6) \qquad \left\langle d^*l_s\left(v_1^{j_1}\cdots v_s^{-1}\right), \lambda_{k_1}\cdots\lambda_{k_{s-1}}\right\rangle = \left\langle l_s\left(v_1^{j_1}\cdots v_s^{-1}\right), \lambda_{-1}\lambda_{k_1}\cdots\lambda_{k_{s-1}}\right\rangle$$
$$+ \left\langle l_s\left(v_1^{j_1}\cdots v_s^{-1}\right), \lambda_{k_1}\cdots\lambda_{k_{s-1}}\lambda_{-1}\right\rangle.$$

But Lemma 2.5 shows that the first term on the right of (8.6) is zero. So the right-hand sides of (8.5) and (8.6) agree, and (8.4) is proved in the case $j_s = -1$. If $j_s \neq -1$ the proof is similar; both sides of (8.4) are zero. So the proposition is proved.

We now assemble the maps l_s $(s \geqslant 0)$ to obtain $l\colon \Gamma^+ F_2 \to \Lambda^*$. Combining Propositions 8.1 and 8.2 we have

PROPOSITION 8.3. $l\colon \Gamma^+ F_2 \to \Lambda^*$ *is an isomorphism of chain complexes.*

We now wish to give $\Gamma^+ F_2$ the structure of a differential coalgebra, and to prove l an isomorphism of differential coalgebras. We observed in §2 that $\Gamma^+ F_2 \equiv \Gamma^+$ is not a subcoalgebra of Γ. Nevertheless, it is a quotient coalgebra of Γ. We show this in the following way.

Consider the coalgebra morphism $k\colon \Gamma \to \Theta^*$ constructed in §7. Composing k with the projection $\Theta^* \to \Lambda^*$ we obtain a coalgebra morphism $\bar{k}\colon \Gamma \to \Lambda^*$. But the restriction of $\bar{k}$ to Γ^+ is the isomorphism $l\colon \Gamma^+ \to \Lambda^*$. Hence if we define Γ^- to be the vector space kernel

$$(8.7) \qquad \Gamma^- = \ker(\bar{k}\colon \Gamma \to \Lambda^*)$$

we get an internal direct sum decomposition:

$$(8.8) \qquad \Gamma = \Gamma^+ \oplus \Gamma^-.$$

On the other hand, $\bar{k}$ is a map of coalgebras, so Γ^- is a two-sided coideal of Γ. Hence, by identifying Γ^+ with the quotient Γ/Γ^- we obtain a coalgebra structure on Γ^+, and since $\bar{k}$ is a coalgebra morphism, so is $l\colon \Gamma^+ \to \Lambda^*$. Finally we observe that Λ^* is a differential coalgebra, and since l is an isomorphism that preserves differential and coproduct, then Γ^+ must also be a differential coalgebra. We summarize:

THEOREM 8.4. $l\colon \Gamma^+ \to \Lambda^*$ *is an isomorphism of differential coalgebras.*

This is Theorem 1.2 of the Introduction.

Although there is a nice formula that determines the coproduct on Γ (Madsen's formula (2.14) above), there seems to be no equally simple description of the coproduct on Γ^+. Although Madsen's formula is still valid in Γ^+, we can no longer claim that by itself it determines the coproduct. This is because when we identify Γ_s^+ with Γ_s/Γ_s^- there is no longer any algebra structure on Γ_s^+, and there is no sense in which the elements $Q_{s,i}$ can be regarded as generators.

Fortunately, it is not hard to compute the coproduct in Γ^+ on any particular element one may choose. This is because a description of Γ^- is available that is more explicit than (8.7), enabling one to compute the direct sum decomposition (8.8) in any special case. In fact, let Δ_s^- be the F_2-subspace of Δ_s defined by

$$\Delta_s^- = \mathrm{Span}\left\{ v_1^{i_1} \cdots v_s^{i_s} \mid i_j < 0 \text{ for at least one } j \right\}.$$

Then it follows easily from (7.1) and (8.7) that

(8.9) $$\Gamma_s^- = \Gamma_s \cap \Delta_s^-.$$

Notice that the fact that Γ^- is a coideal of Γ follows directly from (8.9). In fact it is clear from (2.13) that $\Delta^- = \bigoplus_{s \geqslant 0}(\Delta_s^-)$ is a coideal of Δ.

9. Residue theory.

In this section we study a little further the differential $\partial\colon \Gamma_s^+ \to \Gamma_{s-1}^+$ that comes from identifying Γ_s^+ and Γ_{s-1}^+ with the chain groups $(\Gamma^+ F_2)_s$ and $(\Gamma^+ F_2)_{s-1}$, respectively. This differential is the restriction of the map $\partial\colon \Delta_s \to \Delta_{s-1}$ defined by (8.2), and we are going to show that (8.2) is a kind of "residue". Roughly speaking, to compute $\partial\gamma$, one regards γ as a rational function in $t_1, t_2, \ldots, t_s$, and expands γ in a Laurent series about the origin of the t_s-plane. Then $\partial\gamma$ is the coefficient of t_s^{-1}.

We proceed to make this idea precise. Let R be any commutative ring with unit; and t an indeterminate. Start with the power series ring $R[[t]]$ and invert the elements of the multiplicative subset $\{1, t, t^2, \ldots\}$. Denote the resulting localization by $R[[t]]'$. Then a typical element of $R[[t]]'$ is a formal sum $\sum_{-\infty}^{\infty} r_k t^k$ in which $r_k = 0$ for all k sufficiently negative. The key property of $R[[t]]'$ is given by

LEMMA 9.1. *Let* $f(t) = \sum_{k=0}^{n} a_k t^k$ *be a nonzero element in the polynomial ring* $R[t]$. *Let* k_0 *be the least integer for which* $a_{k_0} \neq 0$. *If* a_{k_0} *is invertible in* R, *then* $f(t)$ *is invertible in* $R[[t]]'$.

This is well known.

We define a homomorphism of R modules res: $R[[t]]' \to R$ by

(9.1) $$\mathrm{res} \sum_{-\infty}^{\infty} r_k t^k = r_{-1}.$$

We can use this construction to define a residue map from Δ_s to Δ_{s-1}. Observe that V_s can be written as a polynomial

(9.2) $$V_s = \sum_{k=1}^{2^s-1} a_k t_s^k$$

with coefficients a_k in $V_{s-1} \subset \Delta_{s-1}$; further, $a_1 = V_1 V_2 \cdots V_{s-1}$ is invertible in Δ_{s-1}. It follows from Lemma 9.1 that V_s is invertible in $\Delta_{s-1}[[t_s]]'$. Consequently there is a unique inclusion i_s of Δ_{s-1}-algebras from $\Delta_s = \Delta_{s-1}[V_s^{\pm}]$ into $\Delta_{s-1}[[t_s]]'$ that sends V_s to the polynomial (9.2). We compose this inclusion with the residue map of (9.1) to obtain Res: $\Delta_s \to \Delta_{s-1}$:

(9.3)

$$\Delta_s = \Delta_{s-1}\left[V_s^{\pm 1}\right] \xrightarrow{\;\;i_s\;\;} \Delta_{s-1}[[t_s]]'$$
$$\searrow_{\mathrm{Res}} \qquad \downarrow \mathrm{res}$$
$$\Delta_{s-1}$$

The main result of this section is

PROPOSITION 9.2. *Let* $\partial \colon \Delta_s \to \Delta_{s-1}$ *be as given in* (8.2). *Then* $\partial = \mathrm{Res} \colon \Delta_s \to \Delta_{s-1}$.

PROOF. Induct on s. If $s = 1$ it is clear. Suppose the result proved for $s - 1$, for some $s \geqslant 2$. Note that Δ_{s-1} is a subalgebra of Δ_s, so that both Δ_{s-1} and Δ_s can be regarded as modules over Δ_{s-1}. One sees easily that both Res and ∂ are Δ_{s-1}-linear maps, so it suffices to check that Res and ∂ agree on the elements $\{v_s^n \mid -\infty < n < \infty\}$ which form a Δ_{s-1} basis for Δ_s. To this end, define an algebra homomorphism $\alpha \colon \Phi_{s-1} \to \Phi_s$ (with Φ_s defined as in (2.7)) by

$$\alpha(t_j) = t_{j+1} + t_{j+1}^2/t_1.$$

It is easy to check that $\alpha(v_j) = v_{j+1}$ for all j, $1 \leqslant j \leqslant s - 1$. In particular, $\alpha(\Delta_{s-1}) \subseteq \Delta_s$. We can define an algebra homomorphism $\alpha' \colon \Delta_{s-2}[[t_{s-1}]]' \to \Delta_{s-1}[[t_s]]'$ in such a way that the following diagram commutes:

$$(9.4) \qquad
\begin{array}{ccccc}
\Delta_{s-1} & \overset{i_{s-1}}{\to} & \Delta_{s-2}[[t_{s-1}]]' & \overset{\mathrm{res}}{\to} & \Delta_{s-2} \\
\alpha \downarrow & & \downarrow \alpha' & & \downarrow \alpha \\
\Delta_s & \overset{i_s}{\to} & \Delta_{s-1}[[t_s]]' & \overset{\mathrm{res}}{\to} & \Delta_{s-1}
\end{array}$$

In fact, the left-hand square commutes if we set $\alpha'(v_j) = v_{j+1}$ $(1 \leqslant j \leqslant s - 2)$ and $\alpha'(t_{s-1}) = t_s(1 + t_s/v_1)$. So we take this as the definition. In particular,

$$(9.5) \qquad \alpha'(t_{s-1}^k) = t_s^k \sum_{j=0}^{\infty} \binom{k}{j} (t_s/v_1)^j$$

for $-\infty < k < \infty$, where we interpret the binomial coefficients as in [8, §2]. Now the commutativity of the right-hand square in (9.4) follows from (9.5), since $\binom{k}{-k-1}$ is nonzero modulo 2 if and only if $k = -1$. But now we can use (9.4) to prove that $\partial(v_s^n) = \mathrm{Res}(v_s^n)$ for all integers n. In fact, (9.4) implies

$$(9.6) \qquad \mathrm{Res}(v_s^n) = \mathrm{res} \circ i_s \circ \alpha(v_{s-1}^n) = \alpha \circ \mathrm{res} \circ i_{s-1}(v_{s-1}^n)$$
$$= \alpha \circ \mathrm{Res}(v_{s-1}^n)$$

where we have used the definition (9.3). But by the inductive hypothesis $\mathrm{Res}(v_{s-1}^{-1}) = 1$; $\mathrm{Res}(v_{s-1}^n) = 0$ if $n \neq -1$. So (9.6) gives $\mathrm{Res}(v_s^{-1}) = 1$; $\mathrm{Res}(v_s^n) = 0$ if $n \neq -1$. So $\mathrm{Res}(v_s^n) = \partial(v_s^n)$, and we are done.

This last result implies Theorem 1.4 of the Introduction.

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